TILTING BUNDLES AND THE "MISSING PART" ON THE WEIGHTED PROJECTIVE LINE OF TYPE $(2, 2, n)$

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ABSTRACT. This paper classifies all the tilting bundles in the category of coherent sheaves on the weighted projective line of weight type $(2, 2, n)$, and investigates the abelianness of the "missing part" from the category of coherent sheaves to the category of finitely generated right modules on the associated tilted algebra for each tilting bundle.

1. INTRODUCTION

Tilting theory arises from the representation theory of finite dimensional algebras and has proved to be a universal method for constructing equivalences between categories, (see for instance [1, 5]). Let A be a finite dimensional algebra and $_AT$ be a tilting left A-module. Happel and Ringel [5] show that $_AT$ gives rise to a torsion pair $(\mathcal{T}(_AT); \mathcal{F}(_AT))$ in the category $\mathrm{mod}\, A$ of finitely generated left A-modules and a corresponding torsion pair $(\mathcal{T}(T_B); \mathcal{F}(T_B))$ in the category of $\mathrm{mod}\, B^{op}$ of finitely generated right B-modules, where $B = \mathrm{End}_A(T)$ is the endomorphism algebra of $_AT$. By Brenner-Butler Theorem [1], the functor $\mathrm{Hom}_A(T, -)$ induces an equivalence between the categories $\mathcal{T}(_AT)$ and $\mathcal{T}(T_B)$, and the functor $\mathrm{Ext}_A^1(T, -)$ induces an equivalence between the category $\mathcal{F}(_AT)$ and $\mathcal{F}(T_B)$. Moreover, if A is hereditary, the torsion pair $(\mathcal{T}(T_B); \mathcal{F}(T_B))$ is splitting, i.e. any indecomposable right B-module is either in $\mathcal{T}(T_B)$ or in $\mathcal{F}(T_B)$.

Geigle and Lenzing [4] extended the notion of tilting module to tilting sheaf in the category $\mathrm{coh}\, \mathbb{X}$ of coherent sheaves on the weighted projective line $\mathbb{X}$. They showed that $T_{\mathrm{can}} = \bigoplus_{0 \le \vec{x} \le \vec{c}} \mathcal{O}(\vec{x})$ is a (canonical) tilting sheaf in $\mathrm{coh}\, \mathbb{X}$, with endomorphism algebra $\Lambda = \mathrm{End}_{\mathrm{coh}\, \mathbb{X}}(T_{\mathrm{can}})$ the canonical algebra of type $(2, 2, n)$. The tilting sheaf T_{can} gives rise to torsion pairs $(\mathcal{X}_0, \mathcal{X}_1)$ in $\mathrm{coh}\, \mathbb{X}$ and $(\mathcal{Y}_1, \mathcal{Y}_0)$ in $\mathrm{mod}\Lambda^{op}$ by setting $\mathcal{X}_i \subseteq \mathrm{coh}\, \mathbb{X}$ the full subcategory of all X with $\mathrm{Ext}_{\mathrm{coh}\, \mathbb{X}}^j(T, X) = 0$ for all $j \ne i$, and $\mathcal{Y}_i \subseteq \mathrm{mod}\Lambda^{op}$ the full subcategory of all Y with $\mathrm{Tor}_j^\Lambda(Y, T) = 0$ for all $j \ne i$. Analogous to the main theorem of tilting theory in the module theoretic setting due to [1] and [5], they proved that the functors $\mathrm{Ext}_{\mathrm{coh}\, \mathbb{X}}^i(T, -) : \mathcal{X}_i \to \mathcal{Y}_i$ and $\mathrm{Tor}_i^\Lambda(-, T) : \mathcal{Y}_i \to \mathcal{X}_i$ define equivalences of categories, inverse to each other, for $i = 0, 1$. Moreover, the torsion pair $(\mathcal{Y}_1, \mathcal{Y}_0)$ is splitting in $\mathrm{mod}\Lambda^{op}$.

Combining the results from [4] with further work related to tilting theory in hereditary categories, Lenzing [7, Theorem 3.1] established the tilting theorem in a hereditary abelian Hom-finite category $\mathscr{H}$ with Serre duality and tilting objects. He pointed out that if the endomorphism algebra Λ of a tilting object T in $\mathscr{H}$ is

not hereditary, there must loose some objects from $\mathscr{H}$ to $\mathrm{mod}\,\Lambda^{op}$. Notice that the category $\mathrm{coh}\,\mathbb{X}$ satisfies the conditions of [7, Theorem 3.1]. It is interesting to study the tilting sheaf in $\mathrm{coh}\,\mathbb{X}$ and the structure of the corresponding factor category $\mathscr{C} = \mathrm{coh}\,\mathbb{X}/[\mathcal{X}_0 \cup \mathcal{X}_1]$, here, $[\mathcal{X}_0 \cup \mathcal{X}_1]$ denotes the ideal of all morphisms in $\mathrm{coh}\,\mathbb{X}$ which factor through a finite direct sum of sheaves from $\mathcal{X}_0 \cup \mathcal{X}_1$.

The factor category $\mathscr{C}$ is called the "missing part" from $\mathrm{coh}\,\mathbb{X}$ to $\mathrm{mod}\Lambda^{op}$ in [2]. It is only an additive category in general. Chen, Lin and Ruan [2] focused on the weighted projective line of weight type $(2, 2, n)$, and showed that for the canonical tilting sheaf T_{can}, the corresponding "missing part" $\mathscr{C}_{\mathrm{can}}$ is an abelian category and isomorphic to $\mathrm{mod}(k\overrightarrow{A}_{n-1})$. Moreover, some examples there indicated that the abelianness is not true if the tilting sheaf contains a direct summand of finite length sheaf. In this paper, we extend the result to a more general case. Namely, we investigate the tilting sheaves not containing finite length direct summands–called tilting bundles–in $\mathrm{coh}\,\mathbb{X}$ and discuss the abelianness of the "missing part" $\mathscr{C}$ from $\mathrm{coh}\,\mathbb{X}$ to $\mathrm{mod}\Lambda^{op}$, where the endomorphism algebra Λ of tilting bundle is called tilted algebra in [7]. We classify all the tilting bundles into two types, consisting of line bundles and not all consisting of line bundles. For the former case, we show that each tilting bundle is the canonical one (under grading shift), hence $\mathscr{C}$ is abelian; for the latter case, we give the trichotomy of the form of each tilting bundle, and show that $\mathscr{C}$ is a product of two abelian categories.

The paper is organized as follows. In section 2, we recall the definition of weighted projective line $\mathbb{X}$ of type $(2, 2, n)$ and some well-known results on the category $\mathrm{coh}\,\mathbb{X}$ of coherent sheaves on $\mathbb{X}$. In section 3, we classify all the tilting bundles in $\mathrm{coh}\,\mathbb{X}$. More precisely, if the tilting bundle T is consisting of line bundles, then T can be obtained from the canonical one under grading shift; if else, T can be decomposed into three parts, $T = T^+(E_i) \oplus (\bigoplus_{i \leq k \leq j} E_k) \oplus T^-(E_j)$, see for instance (3.18). In section 4, we show that the "missing part" corresponding to T of the form (3.18) can be decomposed into a product of two abelian categories.

Throughout the paper, let k be an algebraic closed field and $\mathbb{X}$ be a weighted projective line of weight type $(2, 2, n)$ with $n \geqslant 2$. For simplification, we denote $\mathrm{Ext}^i_{\mathrm{coh}\,\mathbb{X}}(-, -)$ by $\mathrm{Ext}^i(-, -)$ for $i \geq 0$.

2. Preliminary

The notion of weighted projective line was introduced by Geigle and Lenzing [4] to give a geometric treatment to canonical algebra which was studied by Ringel [8]. In this section, we introduce basic definitions and properties on the category of coherent sheaves on the weighted projective line of weight type $(2, 2, n)$.

2.1. **Weighted projective line.** Let $p = (p_1, p_2, p_3) = (2, 2, n)$ and $\mathbb{L}$ be the rank one abelian group on generators $\vec{x}_1, \vec{x}_2, \vec{x}_3$ with relations

$$2\vec{x}_1 = 2\vec{x}_2 = n\vec{x}_3 =: \vec{c}.$$

Then $\mathbb{L}$ is an ordered group, and each $\vec{x} \in \mathbb{L}$ can be uniquely written in normal form

$$(2.1) \qquad \vec{x} = \sum_{i=1}^{3} l_i \vec{x}_i + l\vec{c}, \quad \text{where} \quad 0 \leq l_i \leq p_i - 1 \quad \text{and} \quad l \in \mathbb{Z}.$$

In addition, if $\vec{x}$ is in normal form (2.1), one can define

$$\vec{x} \geq 0 \quad \text{if and only if} \quad l \geq 0,$$

then each $\vec{x} \in \mathbb{L}$ satisfies exactly one of the following two possibilities

$$(2.2) \qquad\qquad \vec{x} \geq 0 \quad \text{or} \quad \vec{x} \leq \vec{\omega} + \vec{c},$$

where $\vec{c}$ is called the canonical element and $\vec{\omega} = \vec{c} - \sum_{i=1}^{3} \vec{x}_i$ is called the dualizing element of $\mathbb{L}$.

Denote by S the commutative algebra

$$S = k[X_1, X_2, X_3]/\langle X_3^n - X_2^2 + X_1^2 \rangle \triangleq k[x_1, x_2, x_3].$$

Then S carries $\mathbb{L}$-graded by setting $\deg x_i = \vec{x}_i$ $(i = 1, 2, 3)$, i.e. S has a decomposition $S = \bigoplus_{\vec{x} \in \mathbb{L}} S_{\vec{x}}$, which satisfies $S_{\vec{x}} S_{\vec{y}} \subseteq S_{\vec{x}+\vec{y}}$ and $S_0 = k$. Let $\mathbb{X}$ be the curve corresponding to S and we call it the weighted projective line of weight type $(2, 2, n)$.

2.2. **Coherent sheaves on the weighted projective line** $\mathbb{X}$. In the sense of Serre-Grothendieck-Gabriel [3], the category of coherent sheaves on the weighted projective line $\mathbb{X}$ is defined as the quotient category

$$\mathrm{coh}\,\mathbb{X} = \mathrm{mod}^{\mathbb{L}} S / \mathrm{mod}_0^{\mathbb{L}} S,$$

where $\mathrm{mod}^{\mathbb{L}} S$ is the category of finitely generated $\mathbb{L}$-graded S-modules and $\mathrm{mod}_0^{\mathbb{L}} S$ the full subcategory of $\mathrm{mod}^{\mathbb{L}} S$ consisting of finite dimensional modules. Use the notation $\widetilde{M} \in \mathrm{coh}\,\mathbb{X}$ for $M \in \mathrm{mod}^{\mathbb{L}} S$, and call the process sheafification:

$$q : \mathrm{mod}^{\mathbb{L}} S \to \mathrm{coh}\,\mathbb{X}; \quad M \mapsto \widetilde{M}.$$

It is easy to see that $\widetilde{M}(\vec{x}) = \widetilde{M(\vec{x})}$. Call $\mathcal{O} = \widetilde{S}$ the structure sheaf of $\mathbb{X}$, and $\mathcal{O}(\vec{x})$ a line bundle for $\vec{x} \in \mathbb{L}$. Then by definition, for each $\vec{x}, \vec{y} \in \mathbb{L}$, we have

$$(2.3) \qquad \mathrm{Hom}(\mathcal{O}(\vec{x}), \mathcal{O}(\vec{y})) = S_{\vec{y}-\vec{x}}.$$

In addition, the category $\mathrm{coh}\,\mathbb{X}$ has a decomposition:

$$\mathrm{coh}\,\mathbb{X} = \mathrm{vect}\,\mathbb{X} \bigvee \mathrm{coh}_0\,\mathbb{X},$$

where the full subcategory $\mathrm{vect}\,\mathbb{X}$ (resp. $\mathrm{coh}_0\,\mathbb{X}$) consists of coherent sheaves not having a simple sub-sheaf (resp. of finite length), $\bigvee$ means each indecomposable sheaf is either in $\mathrm{vect}\,\mathbb{X}$ or in $\mathrm{coh}_0\,\mathbb{X}$, and there are no non-zero morphisms from $\mathrm{coh}_0\,\mathbb{X}$ to $\mathrm{vect}\,\mathbb{X}$. The objects in $\mathrm{vect}\,\mathbb{X}$ are called vector bundles. Moreover, all line bundles belong to $\mathrm{vect}\,\mathbb{X}$, and each vector bundle X has a filtration by line bundles

$$0 = X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_r = X,$$

where each factor $L_i = X_i/X_{i-1}$ is a line bundle.

Let $\Delta = [2, 2, n]$ be the Dynkin diagram attached to $\mathbb{X}$ and $\widetilde{\Delta}$ its extended Dynkin diagram. According to [6], the Auslander-Reiten quiver $\Gamma(\mathrm{vect}\,\mathbb{X})$ of $\mathrm{vect}\,\mathbb{X}$ consists of a single standard component of the form $\mathbb{Z}\widetilde{\Delta}$. Moreover, the category $\mathrm{ind}(\Gamma(\mathrm{vect}\,\mathbb{X}))$ of indecomposable vector bundles on $\mathbb{X}$ is equivalent to the mesh category of $\Gamma(\mathrm{vect}\,\mathbb{X})$.

In [4], Geigle and Lenzing showed that the category $\mathrm{coh}\,\mathbb{X}$ is a hereditary, abelian, k-linear, Hom-finite, Noetherian category with Serre duality, i.e.

$$(2.4) \qquad \mathrm{Ext}^1(X, Y) = D\,\mathrm{Hom}(Y, \tau X),$$

where the k-equivalence $\tau : \mathrm{coh}\,\mathbb{X} \to \mathrm{coh}\,\mathbb{X}$ is given by the shift $X \mapsto X(\vec{\omega})$.

2.3. **Tilting sheaf and Grothendieck group.** Recall from [4] that a coherent sheaf T is called tilting in $\mathrm{coh}\,\mathbb{X}$ if the following properties hold:

(1) T is extension-free, that is, $\mathrm{Ext}^1(T, T) = 0$;
(2) T generates $D^b(\mathrm{coh}\,\mathbb{X})$ as a triangulated category, i.e. $D^b(\mathrm{coh}\,\mathbb{X})$ is the smallest triangulated subcategory of $D^b(\mathrm{coh}\,\mathbb{X})$ containing T;
(3) $\mathrm{gl.dim}(\mathrm{End}(T)) < \infty$.

Geigle and Lenzing [4] showed that condition (3) is a consequence of (1) and (2), and to prove condition (2) it is sufficient to show that $\mathrm{coh}\,\mathbb{X}$ is the smallest abelian subcategory of $\mathrm{coh}\,\mathbb{X}$ containing T since $\mathrm{coh}\,\mathbb{X}$ is hereditary. Moreover, they gave a canonical tilting sheaf $T_{\mathrm{can}} = \bigoplus_{0 \leqslant \vec{x} \leqslant \vec{c}} \mathcal{O}(\vec{x})$ in $\mathrm{coh}\,\mathbb{X}$ whose endomorphism algebra $\Lambda = \mathrm{End}(T_{\mathrm{can}})$ is the canonical algebra of type $(2, 2, n)$.

Let $K_0(\mathbb{X})$ be the Grothendieck group of $\mathrm{coh}\,\mathbb{X}$ and we still write $X \in K_0(\mathbb{X})$ for the class $X \in \mathrm{coh}\,\mathbb{X}$. Then the classes $\mathcal{O}(\vec{x})$ for $0 \leq \vec{x} \leq \vec{c}$ form a $\mathbb{Z}$-basis of $K_0(\mathbb{X})$. There are two additive functions on $K_0(\mathbb{X})$ called rank and degree respectively. The rank function $\mathrm{rk} : K_0(\mathbb{X}) \to \mathbb{Z}$ is characterized by

$$\mathrm{rk}(\mathcal{O}(\vec{x})) = 1 \quad \text{for} \quad \vec{x} \in \mathbb{L},$$

and the degree function $\deg : K_0(\mathbb{X}) \to \mathbb{Z}$ is uniquely determined by setting

$$\deg(\mathcal{O}(\vec{x})) = \delta(\vec{x}) \quad \text{for} \quad \vec{x} \in \mathbb{L},$$

where $\delta : \mathbb{L} \to \mathbb{Z}$ is the group homomorphism defined on generators by

$$\delta(\vec{x}_1) = \delta(\vec{x}_2) = \frac{\mathrm{l.c.m.}(2, n)}{2} \quad \text{and} \quad \delta(\vec{x}_3) = \frac{\mathrm{l.c.m.}(2, n)}{n}.$$

For each $X \in \mathrm{coh}\,\mathbb{X}$, define the slope of X as

$$\mu X = \frac{\deg X}{\mathrm{rk}\, X}.$$

It is an element in $\mathbb{Q} \cup \{\infty\}$. According to [4], each vector bundle has positive rank, then the slope belongs to $\mathbb{Q}$; each object in $\mathrm{coh}_0 \mathbb{X}$ has rank 0, then the slope is ∞.

In [7], Lenzing proved that each indecomposable vector bundle X is exceptional in $\mathrm{coh}\,\mathbb{X}$, that is, X is extension-free and $\mathrm{End}(X) = k$. Moreover, for any two indecomposable vector bundles X and Y, $\mathrm{Hom}(X, X') \neq 0$ implies $\mu X \leq \mu X'$.

3. CLASSIFICATION OF TILTING BUNDLES

In this section, we investigate the tilting bundle in $\mathrm{coh}\,\mathbb{X}$. We make discussion in two cases according to whether it is consisting of line bundles or not. Finally we give a classification of all the tilting bundles.

3.1. Tilting bundle consisting of line bundles. In [4], we know that

$$T_{\mathrm{can}} = \bigoplus_{0 \leq \vec{x} \leq \vec{c}} \mathcal{O}(\vec{x})$$

is a canonical tilting sheaf in $\mathrm{coh}\,\mathbb{X}$. In this subsection, we show that it is the unique tilting bundle in $\mathrm{coh}\,\mathbb{X}$ consisting of line bundles up to twist, that is, each titling bundle consisting of line bundles has the form

$$T_L = \bigoplus_{0 \leq \vec{x} \leq \vec{c}} L(\vec{x}), \text{ for some line bundle } L.$$

Lemma 3.1. *For any line bundle L and $\vec{x} \in \mathbb{L}$, $L \oplus L(\vec{x})$ is extension-free if and only if*

$$-\vec{c} \leq \vec{x} \leq \vec{c}.$$

In particular, if additional $\delta(\vec{x}) \geq 0$, then $\vec{x}$ satisfies one of the following conditions:

(3.1) *(i)* $0 \leq \vec{x} \leq \vec{c}$; *(ii)* $\vec{x} = \vec{x}_1 - \vec{x}_2$; *(iii)* $\vec{x} = \vec{x}_i - k\vec{x}_3$, *for* $i = 1, 2$, $0 < k \leq \dfrac{n}{2}$.

Proof. For any $\vec{x}, \vec{y} \in \mathbb{L}$, by Serre duality (2.4), we have

$$(3.2) \qquad \operatorname{Ext}^1(L(\vec{y}), L(\vec{x})) = D \operatorname{Hom}(L(\vec{x}), L(\vec{\omega} + \vec{y})) = DS_{\vec{\omega} + \vec{y} - \vec{x}}.$$

Hence by (2.2)

$$\operatorname{Ext}^1(L, L(\vec{x})) = 0 \text{ if and only if } \vec{\omega} - \vec{x} \le \vec{\omega} + \vec{c}, \text{ that is, } \vec{x} \ge -\vec{c};$$

and

$$\operatorname{Ext}^1(L(\vec{x}), L) = 0 \text{ if and only if } \vec{\omega} + \vec{x} \le \vec{\omega} + \vec{c}, \text{ that is, } \vec{x} \le \vec{c}.$$

Thus, $L \oplus L(\vec{x})$ is extension-free if and only if

$$(3.3) \qquad -\vec{c} \le \vec{x} \le \vec{c}.$$

In particular, if additional

$$(3.4) \qquad \delta(\vec{x}) \ge 0,$$

we write $\vec{x} = \sum_{i=1}^{3} l_i \vec{x}_i + l\vec{c}$ in normal form and consider all the possibilities of $\vec{x}$ satisfying both conditions (3.3) and (3.4) according to the number m of non-zero coefficients of $\vec{x}_i$ for $1 \le i \le 3$.

Case 1: $m = 0$, then $\vec{x} = 0$ of $\vec{c}$.

Case 2: $m = 1$, then $\vec{x} = l_i \vec{x}_i$ for some $1 \le i \le 3$ and $0 < l_i < p_i$, i.e., $0 < \vec{x} < \vec{c}$.

Case 3: $m = 2$, then $\vec{x} = \vec{x}_1 + \vec{x}_2 - \vec{c}$ or $\vec{x}_i + l_3 \vec{x}_3 - \vec{c}$ for $i = 1, 2$ and $\frac{n}{2} \le l_3 < n$.
That is, $\vec{x} = \vec{x}_1 - \vec{x}_2$ or $\vec{x}_i - k\vec{x}_3$ for $i = 1, 2$ and $0 < k \le \frac{n}{2}$.

Case 4: $m = 3$, none of $\vec{x}$ satisfied.

Summarize up, (3.1) holds. This finishes the proof. $\qquad \square$

The following is the main result in this subsection.

Theorem 3.2. *Let T be a tilting bundle in $\operatorname{coh} \mathbb{X}$ consisting of line bundles. Then*

$$T = \bigoplus_{0 \le \vec{x} \le \vec{c}} L(\vec{x}), \text{ for some line bundle } L.$$

Proof. Let L be a direct summand of T with minimal slope, and $T = \bigoplus_{\vec{x} \in I} L(\vec{x})$ for some finite set I. By [4], T_{can} is a tilting sheaf in $\operatorname{coh} \mathbb{X}$ with $n + 3$ many indecomposable direct summands. It follows that the order

$$(3.5) \qquad |I| = n + 3.$$

For each $\vec{x} \in I$, by Lemma 3.1, $\vec{x}$ satisfies one of the conditions of (3.1). We claim the condition (iii) there doesn't hold. Otherwise, there exist some $0 < k \le \frac{n}{2}$ and $i = 1$ or 2, such that $\vec{x}_i - k\vec{x}_3 \in I$. Assume $k_{\min} = a$, $k_{\max} = b$. Then for any $l \ge n - b + 1$, by (3.2),

$$\operatorname{Ext}^1(L(l\vec{x}_3), L(\vec{x}_i - b\vec{x}_3)) = DS_{(l+b)\vec{x}_3 - \vec{x}_i + \vec{\omega}} \ne 0,$$

since $(l+b)\vec{x}_3 - \vec{x}_i + \vec{\omega} = \vec{x}_1 + \vec{x}_2 - \vec{x}_i + (l + b - 1)\vec{x}_3 - \vec{c} \ge \vec{x}_1 + \vec{x}_2 - \vec{x}_i > 0$. Hence $l\vec{x}_3 \notin I$. Thus by Lemma 3.1, each element $\vec{x}$ from I satisfies one of the following:

$$(i) \ 0 \le \vec{x} \le (n-b)\vec{x}_3; \quad (ii) \ \vec{x} = \vec{x}_1 - \vec{x}_2; \quad (iii) \ \vec{x} = \vec{x}_i - k\vec{x}_3, \ a \le k \le b.$$

It follows that

$$|I| \le (n - b + 1) + 1 + (b - a + 1) = n + 3 - a < n + 3,$$

a contradiction to (3.5). This finishes the claim. Therefore,

$$(3.6) \qquad I \subseteq \{\vec{x} \in \mathbb{L} | 0 \le \vec{x} \le \vec{c} \quad \text{or} \quad \vec{x} = \vec{x}_1 - \vec{x}_2\}.$$

Moreover, for any $k \ge 1$,

$$\operatorname{Ext}^1(L(k\vec{x}_3), L(\vec{x}_1 - \vec{x}_2)) = DS_{(k-1)\vec{x}_3} \ne 0.$$

Combining with (3.5) and (3.6),

$$I = \{\vec{x} \in \mathbb{L} \mid 0 \le \vec{x} \le \vec{c}\}.$$

That is,

$$T = \bigoplus_{0 \le \vec{x} \le \vec{c}} L(\vec{x}).$$

This finishes the proof.   □

3.2. Tilting bundle not all consisting of line bundles.

In this section, we classify the tilting bundles not all consisting of line bundles. Notice that if $n = 2$, then all the indecomposable direct summands of such a tilting bundle form a slice in the Auslander-Reiten quiver of $\operatorname{vect} \mathbb{X}$, hence the classification is obvious. Thus we only consider the case $n \ge 3$. We decompose such a tilting bundle into three parts with respect to rank two indecomposable direct summands of minimal and maximal length.

3.2.1. *Slice in the category of vector bundles.* According to [6], the Auslander-Reiten quiver $\Gamma(\operatorname{vect} \mathbb{X})$ of $\operatorname{vect} \mathbb{X}$ consists of a single standard component having the form $\mathbb{Z}\widetilde{\Delta}$, where $\widetilde{\Delta}$ is the extended Dynkin diagram with associated Dynkin diagram $\Delta = [2, 2, n]$. Moreover, the category $\operatorname{ind}(\Gamma(\operatorname{vect} \mathbb{X}))$ of indecomposable vector bundles on $\mathbb{X}$ is equivalent to the mesh category of $\Gamma(\operatorname{vect} \mathbb{X})$. Furthermore, each indecomposable vector bundle has rank one or two.

For each indecomposable vector bundle X, denote by $\mathcal{S}(X \to)$ (resp. $\mathcal{S}(\to X)$) the slice beginning from X (resp. ending to X) in $\Gamma(\operatorname{vect} \mathbb{X})$, for the definition of slice we refer to [8]. More precisely,

$$\mathcal{S}(X \to) = \{Y \in \operatorname{ind}(\operatorname{vect} \mathbb{X}) \mid \operatorname{Hom}(X, Y) \ne 0 \text{ and } \operatorname{Hom}(X, \tau^m Y) = 0 \text{ for } m \ge 1\},$$

and

$$\mathcal{S}(\to X) = \{Y \in \operatorname{ind}(\operatorname{vect} \mathbb{X}) \mid \operatorname{Hom}(Y, X) \ne 0 \text{ and } \operatorname{Hom}(\tau^{-m} Y, X) = 0 \text{ for } m \ge 1\}.$$

The following lemma plays an important role in classifying the tilting bundles which are not all consisting of line bundles in $\operatorname{coh} \mathbb{X}$.

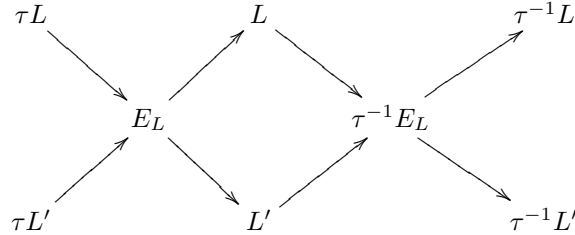**Lemma 3.3.** *Let E and X be two indecomposable vector bundles with $\operatorname{rk} E = 2$.*
  (1) *If $\operatorname{Hom}(E, X) \ne 0$, then $\operatorname{Hom}(E, \tau^{-1} X) \ne 0$.*
  (2) *If $\operatorname{Hom}(X, E) \ne 0$, then $\operatorname{Hom}(\tau X, E) \ne 0$.*

Proof. It suffices to prove statement (1), since the arguments for statement (2) are dual. For contradiction, we assume X is of minimal slope satisfying

$$\operatorname{Hom}(E, X) \ne 0 \text{ but } \operatorname{Hom}(E, \tau^{-1} X) = 0.$$

Concerning the rank of X, we consider the following two cases.

Case 1: $\operatorname{rk} X = 1$, then X is a line bundle L. In the following sub-quiver of $\Gamma(\operatorname{vect} \mathbb{X})$, set $L' = L(\vec{x}_1 - \vec{x}_2)$, and E_L the Auslander bundle corresponding to L.

We claim that

(3.7) $\operatorname{Hom}(E, L') \ne 0.$

In fact, applying $\mathrm{Hom}(E, -)$ to the Auslander-Reiten sequence

$$0 \to \tau L \to E_L \to L \to 0,$$

we obtain an exact sequence:

$$(3.8) \qquad \mathrm{Hom}(E, E_L) \to \mathrm{Hom}(E, L) \to \mathrm{Ext}^1(E, \tau L).$$

By assumption, $\mathrm{Hom}(E, L) \neq 0$, which implies $\mu E < \mu L$. It follows that $\mathrm{Ext}^1(E, \tau L) = D\,\mathrm{Hom}(L, E) = 0$. Hence

$$(3.9) \qquad \mathrm{Hom}(E, E_L) \neq 0.$$

Now by applying $\mathrm{Hom}(E, -)$ to the Auslander-Reiten sequence

$$0 \to \tau L' \to E_L \to L' \to 0,$$

we obtain an exact sequence:

$$0 \to \mathrm{Hom}(E, \tau L') \to \mathrm{Hom}(E, E_L) \to \mathrm{Hom}(E, L').$$

If $\mathrm{Hom}(E, \tau L') = 0$, then (3.7) follows from (3.9); if $\mathrm{Hom}(E, \tau L') \neq 0$, then by the minimality of L, (3.7) also holds; this finishes the claim. Then by applying $\mathrm{Hom}(E, -)$ to the injective map $L' \rightarrowtail L'(\vec{x}_3)$, it follows from (3.7) that $\mathrm{Hom}(E, \tau^{-1}L) = \mathrm{Hom}(E, L'(\vec{x}_3)) \neq 0$, a contradiction.

Case 2: $\mathrm{rk}\,X = 2$, then there exists the following Auslander-Reiten sequence

$$0 \to X \to \oplus_i Y_i \to \tau^{-1}X \to 0.$$

Since $\mathrm{Hom}(E, X) \neq 0$, there exists some i, such that $\mathrm{Hom}(E, Y_i) \neq 0$. Moreover, $\mathrm{rk}\,Y_i \leq 2 = \mathrm{rk}(\tau^{-1}X)$ implies the irreducible map $Y_i \to \tau^{-1}X$ is injective. It follows that $\mathrm{Hom}(E, \tau^{-1}X) \neq 0$, a contradiction. $\qquad\square$

More general, we have

Corollary 3.4. *Let E and X be two indecomposable vector bundles with $\mathrm{rk}\,E = 2$. Then for any $m \geq 0$,*

    (1) $\mathrm{Hom}(E, X) \neq 0$ *implies* $\mathrm{Hom}(E, \tau^{-m}X) \neq 0$;
    (2) $\mathrm{Hom}(X, E) \neq 0$ *implies* $\mathrm{Hom}(\tau^m X, E) \neq 0$.

Remark 3.5. *As a consequence, the expression of the slices with respect to rank two bundle E can be simplified as follows.*

$$(3.10) \quad \mathcal{S}(E \to) = \{Y \in \mathrm{ind}(\mathrm{vect}\,\mathbb{X})|\,\mathrm{Hom}(E, Y) \neq 0 \text{ and } \mathrm{Hom}(E, \tau Y) = 0\},$$

and

$$(3.11) \quad \mathcal{S}(\to E) = \{Y \in \mathrm{ind}(\mathrm{vect}\,\mathbb{X})|\,\mathrm{Hom}(Y, E) \neq 0 \text{ and } \mathrm{Hom}(\tau^{-1}Y, E) = 0\}.$$

By considering the tilting bundle containing rank two indecomposable direct summand, we have the following important observation:
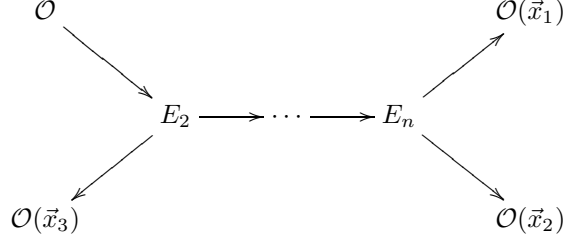
Lemma 3.6. *Let E be an indecomposable vector bundle with $\mathrm{rk}\,E = 2$. Then each tilting object in $\mathrm{coh}\,\mathbb{X}$ contains at most one member from τ-orbit of E.*

Proof. For contradiction, we assume there exists a tilting sheaf in $\mathrm{coh}\,\mathbb{X}$ containing $\tau^{m_1}E$ together with $\tau^{m_2}E$ for some $m_1 < m_2$. Then by Corollary 3.4,

$$\mathrm{Ext}^1(\tau^{m_1}E, \tau^{m_2}E) = D\,\mathrm{Hom}(E, \tau^{m_1 - m_2 + 1}E) \neq 0.$$

Hence $\tau^{m_1}E \oplus \tau^{m_2}E$ is not extension-free, a contradiction. $\qquad\square$

3.2.2. *Domains and properties.* Assume $\mathcal{S}(\mathcal{O} \to)$ in $\Gamma(\text{vect}\,\mathbb{X})$ is given by:

$$\mathcal{O} \qquad\qquad\qquad\qquad\qquad\qquad \mathcal{O}(\vec{x}_1)$$

$$E_2 \longrightarrow \cdots \longrightarrow E_n$$

$$\mathcal{O}(\vec{x}_3) \qquad\qquad\qquad\qquad\qquad\qquad \mathcal{O}(\vec{x}_2)$$

Denote by $\mathcal{L}_i$ the τ-orbit of E_i, for $2 \leq i \leq n$, and by $\mathcal{L}_1$ (resp. $\mathcal{L}'_1, \mathcal{L}_{n+1}, \mathcal{L}'_{n+1}$) the τ-orbit of $\mathcal{O}$ (resp. $\mathcal{O}(\vec{x}_3)$, $\mathcal{O}(\vec{x}_1)$, $\mathcal{O}(\vec{x}_2)$).

Definition 3.7. *For each indecomposable vector bundle X, define the* length *of X by*

$$l(X) = \begin{cases} 1, & \text{if } X \in \mathcal{L}_1 \text{ or } \mathcal{L}'_1; \\ i, & \text{if } X \in \mathcal{L}_i \text{ for } 2 \leq i \leq n; \\ n+1, & \text{if } X \in \mathcal{L}_{n+1} \text{ or } \mathcal{L}'_{n+1}. \end{cases}$$
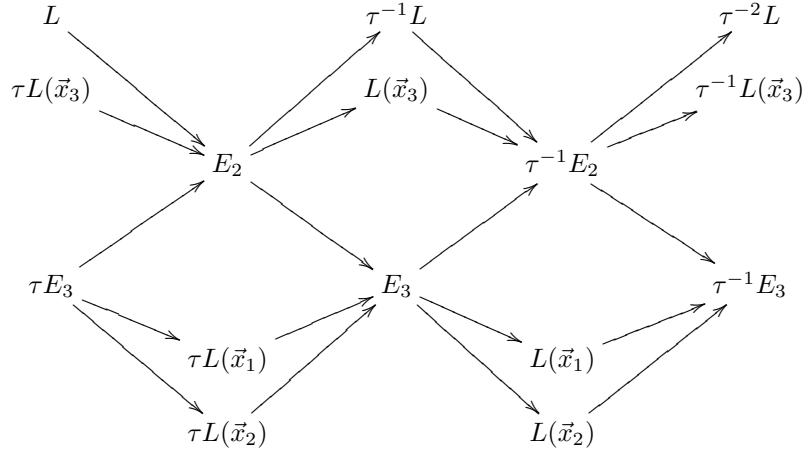
Definition 3.8. *Let X be an indecomposable vector bundle, the* domain *of X, denoted by $\mathrm{Dom}(X)$, is defined as the subset of $\text{vect}\,\mathbb{X}$ consisting of all indecomposable vector bundle Y satisfying that there exist integers $m_1, m_2 \geq 0$, such that $\tau^{m_1} Y \in \mathcal{S}(\to X)$ and $\tau^{-m_2} Y \in \mathcal{S}(X \to)$. In particular, denote by*

$$\mathrm{Dom}^+(X) = \{ Y \in \mathrm{Dom}(X) | l(Y) \leq l(X) \}$$

and

$$\mathrm{Dom}^-(X) = \{ Y \in \mathrm{Dom}(X) | l(Y) \geq l(X) \}.$$

Example 3.9. *Let $\mathbb{X}$ be the weighted projective line of weight type $(2,2,3)$. The following is a sub-quiver of $\Gamma(\text{vect}\,\mathbb{X})$.*
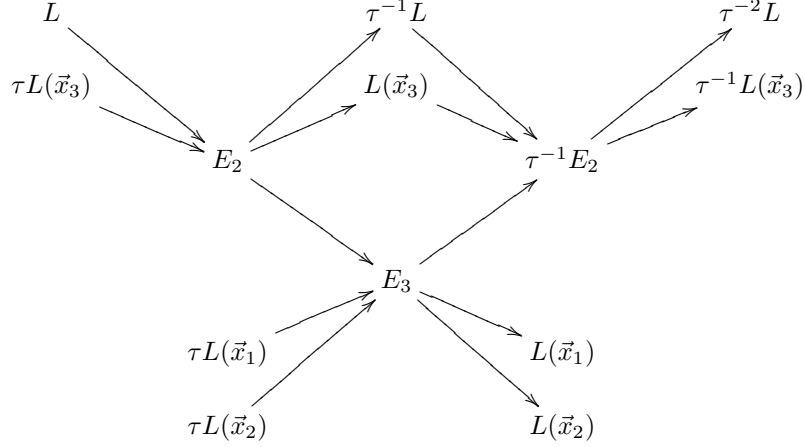

In this picture, we have

$$\mathcal{S}(L \to) = \{ L, L(\vec{x}_1), L(\vec{x}_2), L(\vec{x}_3), E_2, E_3 \};$$

$$\mathcal{S}(E_3 \to) = \{ E_3, \tau^{-1} E_2, L(\vec{x}_1), L(\vec{x}_2), \tau^{-1} L(\vec{x}_3), \tau^{-2} L \};$$

$$\mathcal{S}(\to E_3) = \{ L, \tau L(\vec{x}_1), \tau L(\vec{x}_2), \tau L(\vec{x}_3), E_2, E_3 \};$$

and $\mathrm{Dom}(E_3)$ *has the form below.*

Moreover, $\mathrm{Dom}^+(E_3)$ *(resp.* $\mathrm{Dom}^-(E_3)$*) consists of all the bundles posit on the above (resp. below) of* E_3*, in each case, containing* E_3*.*

Lemma 3.10. *Let* X *and* Y *be two indecomposable vector bundles. Then*

$$Y \in \mathrm{Dom}(X) \quad \text{if and only if} \quad X \in \mathrm{Dom}(Y).$$

Proof. By definition, $Y \in \mathrm{Dom}(X)$ if and only if there exist $m_1, m_2 \geq 0$, such that $\tau^{m_1} Y \in \mathcal{S}(\to X)$ and $\tau^{-m_2} Y \in \mathcal{S}(X \to)$, equivalently, if and only if $X \in \mathcal{S}(\tau^{m_1} Y \to)$ and $X \in \mathcal{S}(\to \tau^{-m_2} Y)$, thus if and only if $\tau^{-m_1} X \in \mathcal{S}(Y \to)$ and $\tau^{m_2} X \in \mathcal{S}(\to Y)$, that is, $X \in \mathrm{Dom}(Y)$. $\qquad\square$

Proposition 3.11. *For any indecomposable vector bundles* X *and* Y*, the following statements are equivalent.*

(1) $X \in \mathrm{Dom}^+(Y)$;
(2) $\mathrm{Dom}^+(X) \subseteq \mathrm{Dom}^+(Y)$;
(3) $Y \in \mathrm{Dom}^-(X)$;
(4) $\mathrm{Dom}^-(Y) \subseteq \mathrm{Dom}^-(X)$.

Proof. Firstly, we show that statements (1) and (2) are equivalent. In fact, if (2) holds, then (1) follows from $X \in \mathrm{Dom}^+(X)$. On the other hand, assume (1) holds, then by definition, there exists some $m_1' \geq 0$, such that $\tau^{m_1'} X \in \mathcal{S}(\to Y)$. For any indecomposable vector bundle $Z \in \mathrm{Dom}^+(X)$, there exists some $m_1'' \geq 0$, such that $\tau^{m_1''} Z \in \mathcal{S}(\to X)$. Let $m_1 = m_1' + m_1''$. Then $m_1 \geq 0$ and $\tau^{m_1} Z \in \mathcal{S}(\to Y)$. Similarly, there exists some $m_2 \geq 0$, such that $\tau^{-m_2} Z \in \mathcal{S}(\to Y)$. Hence $Z \in \mathrm{Dom}^+(Y)$, that is, (2) holds, as claimed.

By dually, we have statements (3) and (4) are equivalent. Moreover, by definition, $X \in \mathrm{Dom}^+(Y)$ if and only if $X \in \mathrm{Dom}(Y)$ and $l(X) \leq l(Y)$, thus if and only if $Y \in \mathrm{Dom}^-(X)$ by Lemma 3.10. That is, statements (1) and (3) are equivalent, this finishes the proof. $\qquad\square$

For rank two indecomposable vector bundles, there is an equivalent description of their domains, related to extension-free and then tilting objects.

Lemma 3.12. *Let* E *and* X *be two indecomposable vector bundles with* $\mathrm{rk}\, E = 2$. *Then* $X \in \mathrm{Dom}(E)$ *if and only if* $E \oplus X$ *is extension-free.*

Proof. If $X \in \mathrm{Dom}(E)$, there exist $m_1, m_2 \geq 0$, such that

$$\tau^{m_1} X \in \mathcal{S}(\to E) \text{ and } \tau^{-m_2} X \in \mathcal{S}(E \to).$$

By (3.10) and (3.11), we have

$$\mathrm{Hom}(\tau^{m_1-1}X, E) = 0 \text{ and } \mathrm{Hom}(E, \tau^{-m_2+1}X) = 0.$$

Then by Corollary 3.4,

$$\mathrm{Hom}(\tau^{-1}X, E) = 0 \text{ and } \mathrm{Hom}(E, \tau X) = 0.$$

Using Serre duality, we get

$$\mathrm{Ext}^1(E, X) = 0 \text{ and } \mathrm{Ext}^1(X, E) = 0.$$

That is, $E \oplus X$ is extension-free.

The sufficiency follows by similar considerations, by going the steps of the preceding proof backwards. $\qquad\square$

More general, we have

Lemma 3.13. *Let E be an indecomposable vector bundle of rank two. Then for any indecomposable objects $X \in \mathrm{Dom}^+(E)$ and $Y \in \mathrm{Dom}^-(E)$, $X \oplus Y$ is extension-free.*

Proof. By Proposition 3.11, $X \in \mathrm{Dom}^+(E)$ implies $E \in \mathrm{Dom}^-(X)$, and then from $Y \in \mathrm{Dom}^-(E)$ we get $Y \in \mathrm{Dom}^-(X)$, equivalently, $X \in \mathrm{Dom}^+(Y)$. Hence, if $\mathrm{rk}\, X = 2$ or $\mathrm{rk}\, Y = 2$, then by Lemma 3.12, $X \oplus Y$ is extension-free. If else, X, Y are both line bundles. Then from the structure of $\mathrm{Dom}(E)$ and by Lemma 3.1, it's easy to see that $X \oplus Y$ is also extension-free, as claimed. $\qquad\square$

3.2.3. *Classification Theorem.* Now we will give our main result of this section. Before giving the classification theorem, we still need some preparations.

Lemma 3.14. *Let E be an indecomposable vector bundle of rank two with $\mathcal{S}(\to E) \cap \mathcal{L}_1 = L$, and $F = \bigoplus_{\vec{x} \in I} L(\vec{x})$ be a direct sum of pair-wise distinct line bundles from $\mathrm{Dom}^+(E)$. If $E \oplus F$ is extension-free, then the order $|I| \leq l(E)$.*

Proof. From the structure of $\Gamma(\mathrm{vect}\, \mathbb{X})$, we know that each line bundle from $\mathrm{Dom}^+(E)$ has the form

$$L(k\vec{x}_3 + \vec{x}_1 - \vec{x}_2) \quad \text{or} \quad L(k\vec{x}_3), \quad \text{for} \quad 0 \leq k \leq l(E) - 1.$$

By symmetry of $\mathcal{L}_1$ and $\mathcal{L}'_1$, without loss of generality, we assume there exists some $0 \leq a \leq l(E) - 1$, such that $L(a\vec{x}_3)$ is a direct summand of F with minimal slope. Then by Lemma 3.1, for each $\vec{x} \in I$,

$$(3.12) \qquad \vec{x} = a\vec{x}_3 + \vec{x}_1 - \vec{x}_2 \quad \text{or} \quad \vec{x} = k\vec{x}_3, \quad \text{for} \quad a \leq k \leq l(E) - 1.$$

Notice from (3.2) that

$$\mathrm{Ext}^1(L(k\vec{x}_3), L(a\vec{x}_3 + \vec{x}_1 - \vec{x}_2)) = DS_{(k-a-1)\vec{x}_3}.$$

Hence, for any $k > a$, $L(k\vec{x}_3) \oplus L(a\vec{x}_3 + \vec{x}_1 - \vec{x}_2)$ is not extension-free. Combining with (3.12), we get $|I| \leq \max\{2, l(E) - a\} \leq l(E)$. $\qquad\square$

Remark 3.15. *Keep the notation in Lemma 3.14. Denote by*

$$E^u = \bigoplus_{0 \leq k \leq l(E) - 1} L(k\vec{x}_3).$$

In particular, if $l(E) = 2$, denote by

$$E^l = L \oplus L(\vec{x}_1 - \vec{x}_2).$$

Now define a set $\mathrm{Br}^+(E)$ as below: if $l(E) > 2$, then $\mathrm{Br}^+(E) = \{E^u, E^u(\vec{x}_1 - \vec{x}_2)\}$; if $l(E) = 2$, then $\mathrm{Br}^+(E) = \{E^l, E^l(\vec{x}_3), E^u, E^u(\vec{x}_1 - \vec{x}_2)\}$. Then according to the proof of Lemma 3.14, we have

$$(3.13) \qquad\qquad |I| = l(E) \quad \text{if and only if} \quad F \in \mathrm{Br}^+(E).$$

Dually, we have the follow lemma and remark.

Lemma 3.16. *Let E be an indecomposable vector bundle of rank two with $\mathcal{S}(E \to) \cap \mathcal{L}_{n+1} = L'$, and $F = \bigoplus_{\vec{x} \in J} L'(\vec{x})$ be a direct sum of pair-wise distinct line bundles from $\mathrm{Dom}^-(E)$. If $E \oplus F$ is extension-free, then $|J| \leq n - l(E) + 2$.*

Remark 3.17. *Keep the notation in Lemma 3.16. Denote by*

$$E^d = \bigoplus_{0 \leq k \leq n - l(E) + 1} L'(-k\vec{x}_3).$$

In particular, if $l(E) = n$, denote by

$$E^r = L' \oplus L'(\vec{x}_1 - \vec{x}_2).$$

Now define a set $\mathrm{Br}^-(E)$ as below: if $l(E) < n$, then $\mathrm{Br}^-(E) = \{E^d, E^d(\vec{x}_1 - \vec{x}_2)\}$; if $l(E) = n$, then $\mathrm{Br}^-(E) = \{E^r, E^r(-\vec{x}_3), E^d, E^d(\vec{x}_1 - \vec{x}_2)\}$. Then in Lemma 3.16, we have

$$(3.14) \qquad |J| = n - l(E) + 2 \quad \text{if and only if} \quad F \in \mathrm{Br}^-(E).$$

For convenience to describe the classification theorem, we need to introduce a conception named sub-slice.

Definition 3.18. *Assume $2 \leq i \leq j \leq n$. Let $E_k \in \mathcal{L}_k$ be indecomposable vector bundles for $i \leq k \leq j$. Then $\{E_k | i \leq k \leq j\}$ is called a sub-slice from E_i to E_j if it can be extended to a slice in $\Gamma(\mathrm{vect}\,\mathbb{X})$.*

Lemma 3.19. *Assume $L \in \mathcal{L}_1$, and $\{E_k | E_k \in \mathcal{L}_k, 2 \leq k \leq n\}$ is a sub-slice from E_2 to E_n contained in the slice $\mathcal{S}(L \to)$. Then for $2 \leq k \leq n - 1$,*

(1) $E_{k+1}^u = E_k^u \oplus L(k\vec{x}_3)$;
(2) $E_k \in \langle E_{k+1}, L(k\vec{x}_3), L((k-1)\vec{x}_3) \rangle$, *the smallest full subcategory of $\mathrm{coh}\,\mathbb{X}$ containing $E_{k+1}, L(k\vec{x}_3)$ and $L((k-1)\vec{x}_3)$ closed under the third term of exact sequence.*

Proof. Assertion (1) directly follows from the definition of E_k^u. For assertion (2), we consider the following exact sequence obtained by induction on k:

$$0 \to L \to E_k \to L(-\vec{\omega} + (k-2)\vec{x}_3) \to 0,$$

which induces a pullback commutative diagram as follows:

Then we obtain an exact sequence:

$$(3.15) \qquad 0 \to E_k \to E_{k+1} \to S_{3,k} \to 0,$$

where $S_{3,k}$ is a simple sheaf concentrated at the point $\vec{x}_3$ determined by the following exact sequence:

$$(3.16) \qquad 0 \to L(-\vec{\omega} + (k-2)\vec{x}_3) \to L(-\vec{\omega} + (k-1)\vec{x}_3) \to S_{3,k} \to 0.$$

Notice that $S_{3,k}(\vec{x}_1 - \vec{x}_2) = S_{3,k}$ and $-\vec{\omega} = -\vec{x}_1 + \vec{x}_2 + \vec{x}_3$. We obtain the following exact sequence obtained from (3.16) by taking grading shift of $\vec{x}_1 - \vec{x}_2$:

$$(3.17) \qquad 0 \to L((k-1)\vec{x}_3) \to L(k\vec{x}_3) \to S_{3,k} \to 0.$$

Combining with (3.15) and (3.17), we finish the proof. $\qquad \square$

By duality, we have

Lemma 3.20. *Assume $L' \in \mathcal{L}_{n+1}$, and $\{E_k | E_k \in \mathcal{L}_k, 2 \leq k \leq n\}$ is a sub-slice from E_2 to E_n contained in the slice $\mathcal{S}(\to L')$. Then for $2 \leq k \leq n-1$,*

(1) $E_k^d = E_{k+1}^d \oplus L'(-(n-k+1)\vec{x}_3)$;
(2) $E_{k+1} \in \langle E_k, L'(-(n-k+1)\vec{x}_3), L'(-(n-k)\vec{x}_3) \rangle$.

Now we give the main result in this section.

Theorem 3.21 (Classification theorem). *Assume T is a bundle in $\operatorname{coh}\mathbb{X}$ not all consisting of line bundles. Then T is tilting if and only if there exist $2 \leq i \leq j \leq n$ and $E_k \in \mathcal{L}_k$ for $i \leq k \leq j$, such that*

$$T = T^+(E_i) \oplus \left(\bigoplus_{i \leq k \leq j} E_k \right) \oplus T^-(E_j), \tag{3.18}$$

where $T^+(E_i) \in \operatorname{Br}^+(E_i)$, $T^-(E_j) \in \operatorname{Br}^-(E_j)$ and $\{E_k | i \leq k \leq j\}$ is a sub-slice from E_i to E_j.

Proof. On one hand, we show that each tilting bundle T in $\operatorname{coh}\mathbb{X}$ not all consisting of line bundles has the form (3.18). Let E_i (resp. E_j) be the rank two indecomposable direct summand of T with minimal (resp. maximal) length. Then by Corollary 3.6, E_i (resp. E_j) is uniquely determined. Moreover, T has a decomposition

$$T = T_1 \oplus T_2 \oplus T_3,$$

where the indecomposable summands of T_1 (resp. T_3) are of length 1 (resp. $n+1$), and $T_2 \in \bigoplus_{i \leq k \leq j} \mathcal{L}_k$. Then by Lemma 3.12, $T_1 \oplus E_i$ is extension-free implies that $T_1 \in \operatorname{Dom}^+(E_i)$, and $T_3 \oplus E_j$ is extension-free implies that $T_3 \in \operatorname{Dom}^-(E_j)$. Thus by Lemma 3.14 and 3.16, we have

$$|T_1| \leq i \quad \text{and} \quad |T_3| \leq n - j + 2, \tag{3.19}$$

where $|T_m|$ denotes the number of pair-wise distinct indecomposable direct summands of T_m. Moreover, Lemma 3.6 implies

$$|T_2| \leq j - i + 1. \tag{3.20}$$

It follows that $|T| = \sum_{i=1}^{3} |T_i| \leq n + 3 = |T|$. Hence each inequality in (3.19) and (3.20) should be equality. Thus $T_1 \in \operatorname{Br}^+(E_i)$ (resp. $T_3 \in \operatorname{Br}^-(E_j)$) by (3.13) (resp. (3.14)), and $T_2 = \bigoplus_{i \leq k \leq j} E_k$ with $E_k \in \mathcal{L}_k$ for any $i \leq k \leq j$. Furthermore, for $i \leq k < j$, $E_k \oplus E_{k+1}$ is extension-free implies that $E_k \in \operatorname{Dom}^+(E_{k+1})$. By the structure of $\Gamma(\operatorname{vect}\mathbb{X})$, there exists an irreducible map between E_k and E_{k+1}. Hence $\{E_k | i \leq k \leq j\}$ is a sub-slice from E_i to E_j.

On the other hand, we show that if T has the form (3.18), then T is a tilting sheaf in $\operatorname{coh}\mathbb{X}$. We only consider the case

$$T = E_i^u \oplus \left(\bigoplus_{i \leq k \leq j} E_k \right) \oplus E_j^d, \tag{3.21}$$

since the proofs for the other choices of $T^+(E_i) \in \operatorname{Br}^+(E_i)$ and $T^-(E_j) \in \operatorname{Br}^-(E_j)$ are quite similar.

Firstly, we claim that T of the form (3.21) is extension-free. In fact, since $\{E_k | i \leq k \leq j\}$ is a sub-slice from E_i to E_j, we know that $\bigoplus_{i \leq k \leq j} E_k$ is a direct summand of a tilting sheaf corresponding to a slice, hence it is extension-free. Meanwhile, Lemma 3.1 implies that both of E_i^u and E_j^d are extension-free. Moreover, notice that $E_i^u \in \operatorname{Dom}^+(E_i)$ and $E_j^d \in \operatorname{Dom}^-(E_j) \subseteq \operatorname{Dom}^-(E_i)$, hence $E_i^u \oplus E_j^d$ is extension-free by Lemma 3.13. Furthermore, for each $i \leq k \leq j$, $E_i^u \in \operatorname{Dom}^+(E_i) \subseteq \operatorname{Dom}^+(E_k)$

implies $E_k \oplus E_i^u$ is extension-free, and $E_j^d \in \mathrm{Dom}^-(E_j) \subseteq \mathrm{Dom}^-(E_k)$ implies $E_k \oplus E_j^d$ is extension-free, as claimed.

Secondly, we remain to prove that $\mathrm{coh}\,\mathbb{X}$ is generated by T. We extend the sub-slice $\{E_k | i \leq k \leq j\}$ to a slice as following,

$$
\begin{array}{ccc}
L & & L' \\
\searrow & & \nearrow \\
& E_2 \to \cdots \to E_i - \cdots - E_k - \cdots - E_j \to \cdots \to E_n & \\
\swarrow & & \nwarrow \\
L(\vec{x}_3) & & L'(-\vec{x}_3)
\end{array}
$$

where $E_m \in \mathcal{L}_m$ for $2 \leq m \leq n$, $L = \mathcal{S}(\to E_i) \cap \mathcal{L}_1$, $L' = \mathcal{S}(E_j \to) \cap \mathcal{L}_{n+1}$, $E_i - \cdots - E_k - \cdots - E_j$ represents the given sub-slice from E_i to E_j and each arrow $E_m \to E_{m+1}$ represents an irreducible injective for $m < i$ and $m \geq j$. Then the direct sum of all the bundles from the slice forms a tilting bundle T' in $\mathrm{coh}\,\mathbb{X}$. By Lemma 3.19, for any $m < i$, $E_m \in \langle E_{m+1}, L(m\vec{x}_3), L((m-1)\vec{x}_3) \rangle$. It follows that $\bigoplus_{m=2}^{i-1} E_m \in \langle E_i, \bigoplus_{a=1}^{i-1} L(a\vec{x}_3) \rangle \subseteq \langle E_i \oplus E_i^u \rangle$. Similarly, by Lemma 3.20, we can get $\bigoplus_{m=j+1}^{n} E_m \in \langle E_j \oplus E_j^d \rangle$. Hence $\mathrm{coh}\,\mathbb{X} = \langle T' \rangle \subseteq \langle T \rangle \subseteq \mathrm{coh}\,\mathbb{X}$, as claimed. $\qquad\square$

4. The structure of the "missing part"

In this section we investigate the "missing part" from $\mathrm{coh}\,\mathbb{X}$ to $\mathrm{mod}\,\Lambda^{op}$ for any tilting bundle T, where $\Lambda = \mathrm{End}(T)$ is the endomorphism algebra of T.

Firstly, we recall the definition of "missing part" from [2]. Let T be a tilting bundle in $\mathrm{coh}\,\mathbb{X}$ with endomorphism algebra Λ. Then T gives rise to torsion pairs $(\mathcal{X}_0, \mathcal{X}_1)$ in $\mathrm{coh}\,\mathbb{X}$ and $(\mathcal{Y}_1, \mathcal{Y}_0)$ in $\mathrm{mod}\,\Lambda^{op}$ by setting

$$\mathcal{X}_i = \{X \in \mathrm{coh}\,\mathbb{X} | \mathrm{Ext}^j(T, X) = 0, j \neq i\}$$

and

$$\mathcal{Y}_i = \{Y \in \mathrm{mod}\,\Lambda^{op} | \mathrm{Tor}_j^{\Lambda}(Y, T) = 0, j \neq i\}.$$

Geigle and Lenzing [4] proved that the functors

$$\mathrm{Ext}^i(T, -) : \mathcal{X}_i \to \mathcal{Y}_i \quad \text{and} \quad \mathrm{Tor}_i^{\Lambda}(-, T) : \mathcal{Y}_i \to \mathcal{X}_i$$

define equivalences of categories, inverse to each other, for $i = 0, 1$. In particular, the torsion pair $(\mathcal{Y}_1, \mathcal{Y}_0)$ is splitting. Hence it's natural to get the following definition:

Definition 4.1 ([2]). *Let T be a tilting bundle in $\mathrm{coh}\,\mathbb{X}$ with endomorphism algebra Λ. The "missing part" from $\mathrm{coh}\,\mathbb{X}$ to $\mathrm{mod}(\Lambda^{op})$ is defined to be the factor category*

$$\mathscr{C} = \mathrm{coh}(\mathbb{X})/[\mathcal{X}_0 \cup \mathcal{X}_1],$$

where $[\mathcal{X}_0 \cup \mathcal{X}_1]$ denotes the ideal of all morphisms in $\mathrm{coh}\,\mathbb{X}$ which factor through a finite direct sum of coherent sheaves from $\mathcal{X}_0 \cup \mathcal{X}_1$.

By definition, the "missing part" $\mathscr{C}$ is an additive category and has the expression

$$(4.1) \qquad \mathscr{C} = \mathrm{add}\{X \in \mathrm{ind}(\mathrm{coh}\,\mathbb{X}) | \mathrm{Hom}(T, X) \neq 0 \neq \mathrm{Ext}^1(T, X)\}.$$

Notice that if T is consisting of line bundles, then by Theorem 3.2, T has the form

$$T_L = \bigoplus_{0 \leq \vec{x} \leq \vec{c}} L(\vec{x}), \quad \text{for some line bundle } L.$$

Hence, by [2], the corresponding "missing part" $\mathscr{C}_L$ has the expression

$$(4.2) \qquad \mathscr{C}_L = \mathrm{add}\{X \in \mathrm{ind}(\mathrm{coh}\,\mathbb{X}) | \mathrm{Hom}(L, X) \neq 0 \neq \mathrm{Ext}^1(L(\vec{c}), X)\},$$

which is an abelian category. Therefore, we only consider the case T not all consisting of line bundles. In particular, for $n = 2$, the indecomposable direct summands of such a tilting bundle form a slice in $\Gamma(\mathrm{vect}\,\mathbb{X})$, whose endomorphism algebra is hereditary, hence nothing is missing from $\mathrm{coh}\,\mathbb{X}$ to $\mathrm{mod}\,\Lambda^{op}$, that is, $\mathscr{C} = 0$. Thus we deduce to the case $n \geq 3$. According to Theorem 4.11, T has the form (3.18). In order to describe the details more precisely, we only consider T of the form

$$(4.3) \qquad T = E_i^u \oplus \big(\bigoplus_{i \leq k \leq j} E_k \big) \oplus E_j^d.$$

The arguments for other choices of $T^+(E_i) \in \mathrm{Br}^+(E_i)$ and $T^-(E_j) \in \mathrm{Br}^-(E_j)$ are quite similar.

The following lemma is crucial in this section, which is useful to give a more explicit description of $\mathscr{C}$.

Lemma 4.2. *Let E_k be an indecomposable rank two summand of T. For any indecomposable vector bundle X,*

   (1) *if $\mathrm{Ext}^1(E_k, X) \neq 0$, then $\mathrm{Hom}(T, X) = 0$;*
   (2) *if $\mathrm{Ext}^1(X, E_k) \neq 0$, then $\mathrm{Ext}^1(T, X) = 0$.*

Proof. We only prove the statement (1), since assertion (2) can be obtained by similar arguments. For contradiction of assertion (1), we assume there exists an indecomposable summand T_i of T satisfying $\mathrm{Hom}(T_i, X) \neq 0$. Then there exists some $m' \geq 0$, such that

$$\mathrm{Hom}(\tau^{-m'} T_i, X) \neq 0 \quad \text{and} \quad \mathrm{Hom}(\tau^{-m'-1} T_i, X) = 0.$$

That is,

$$\tau^{-m'} T_i \in \mathcal{S}(\to X).$$

Since $\mathrm{Ext}^1(E_k, X) \neq 0$, by Serre duality, we have $\mathrm{Hom}(\tau^{-1} X, E_k) \neq 0$. Then by similar arguments, there exists some $m'' \geq 1$, such that

$$\tau^{-m''} X \in \mathcal{S}(\to E_k).$$

It follows that there exists some $m \geq m' + m'' \geq 1$, such that $\tau^{-m} T_i \in \mathcal{S}(\to E_k)$, which implies $T_i \notin \mathrm{Dom}(E_k)$. Therefore, by Lemma 3.12, $T_i \oplus E_k$ is not extension-free, a contradiction. This finishes the proof. $\qquad\square$

Corollary 4.3. *For any indecomposable rank two summand E_k of T, $\mathrm{ind}(\mathscr{C}) \subseteq \mathrm{Dom}(E_k)$.*

Proof. For any indecomposable object $X \in \mathscr{C}$, by (4.1), $\mathrm{Hom}(T, X) \neq 0 \neq \mathrm{Ext}^1(T, X)$. Then by Lemma 4.2, we have $\mathrm{Ext}^1(E_k, X) = 0 = \mathrm{Ext}^1(X, E_k)$. That is, $E_k \oplus X$ is extension-free. Hence by Lemma 3.12, $X \in \mathrm{Dom}(E_k)$, as claimed. $\qquad\square$

The following lemma shows that the "missing part" $\mathscr{C}$ contains two components.

Lemma 4.4. *The "missing part" $\mathscr{C}$ has a decomposition*

$$\mathscr{C} = \mathscr{C}_1 \coprod \mathscr{C}_2,$$

where

$$\mathrm{ind}(\mathscr{C}_1) = \mathrm{ind}(\mathscr{C}) \cap \mathrm{Dom}^+(E_i) \quad \text{and} \quad \mathrm{ind}(\mathscr{C}_2) = \mathrm{ind}(\mathscr{C}) \cap \mathrm{Dom}^-(E_j).$$

Proof. Firstly, we claim that

$$(4.4) \qquad \tau^m E_k \notin \mathscr{C}, \quad \text{for any } m \in \mathbb{Z} \text{ and } i \leq k \leq j.$$

In fact, for $i \leq k \leq j$, if $m = 0$, then $\mathrm{Ext}^1(T, E_k) = 0$ implies that $E_k \notin \mathscr{C}$; if $m \neq 0$, then by Corollary 3.4, $\tau^m E_k \notin \mathrm{Dom}(E_k)$. Combining with Lemma 4.3, we get $\tau^m E_k \notin \mathscr{C}$, as claimed. Thus $\mathscr{C}$ has a decomposition

$$\mathscr{C} = \mathscr{C}_1 \coprod \mathscr{C}_2,$$

where

$$\mathscr{C}_1 = \mathrm{add}\{X \in \mathrm{ind}(\mathscr{C}) | l(X) < i\} \quad \text{and} \quad \mathscr{C}_2 = \mathrm{add}\{X \in \mathrm{ind}(\mathscr{C}) | l(X) > j\}.$$

Moreover, (4.4) implies $E_i \notin \mathscr{C}$, hence for any object $X \in \mathrm{ind}(\mathscr{C}) \cap \mathrm{Dom}^+(E_i)$, we have $l(X) < i$. It follows that $\mathrm{ind}(\mathscr{C}) \cap \mathrm{Dom}^+(E_i) \subseteq \mathrm{ind}(\mathscr{C}_1)$. On the other hand, for any indecomposable object $X \in \mathscr{C}_1$, by Corollary 4.3, we have $X \in \mathrm{Dom}^+(E_i)$. It follows that $\mathrm{ind}(\mathscr{C}_1) \subseteq \mathrm{ind}(\mathscr{C}) \cap \mathrm{Dom}^+(E_i)$. Hence $\mathrm{ind}(\mathscr{C}_1) = \mathrm{ind}(\mathscr{C}) \cap \mathrm{Dom}^+(E_i)$. Similarly, we have $\mathrm{ind}(\mathscr{C}_2) = \mathrm{ind}(\mathscr{C}) \cap \mathrm{Dom}^-(E_j)$. This finishes the proof. $\qquad\square$

We are now going to describe the categories $\mathscr{C}_1$ and $\mathscr{C}_2$ more precisely. By symmetric of $\mathscr{C}_1$ and $\mathscr{C}_2$, we only show the results for $\mathscr{C}_1$ in the following. One should keep in mind that all the statements also hold for $\mathscr{C}_2$ by using the duality of $\mathcal{L}_1$ (resp. $\mathcal{L}_1'$) and $\mathcal{L}_{n+1}$ (resp. $\mathcal{L}_{n+1}'$). The following lemma gives an explicit description of $\mathscr{C}_1$, comparing with (4.2).

Lemma 4.5. *Assume* $\mathcal{S}(\to E_i) \cap \mathcal{L}_1 = L$, *then*

$$(4.5) \qquad \mathscr{C}_1 = \mathrm{add}\{X \in \mathrm{ind}(\mathrm{coh}\,\mathbb{X}) | \mathrm{Hom}(L, X) \neq 0 \neq \mathrm{Ext}^1(L((i-1)\vec{x}_3), X)\}.$$

Proof. For any indecomposable sheaf X, if $\mathrm{Hom}(L, X) \neq 0 \neq \mathrm{Ext}^1(L((i-1)\vec{x}_3), X)$, then $X \in \mathscr{C}$ since both of L and $L((i-1)\vec{x}_3)$ are direct summands of T. Moreover, by Lemma 3.13, $\mathrm{Ext}^1(L((i-1)\vec{x}_3), X) \neq 0$ implies $X \notin \mathscr{C}_2$, hence $X \in \mathscr{C}_1$.

On the other hand, for any indecomposable object $X \in \mathscr{C}_1$, we have $X \in \mathrm{Dom}^+(E_k)$ for $i \leq k \leq j$. So by Lemma 3.12, $\mathrm{Ext}^1(E_k, X) = 0$. Moreover, by Lemma 3.13, $\mathrm{Ext}^1(E_j^d, X) = 0$. Hence $\mathrm{Ext}^1(T, X) \neq 0$ implies $\mathrm{Ext}^1(E_i^u, X) \neq 0$. Then there exists some $0 \leq m \leq i-1$, such that $\mathrm{Ext}^1(L(m\vec{x}_3), X) \neq 0$. By applying $\mathrm{Hom}(-, X)$ to the injective morphism $L(m\vec{x}_3) \hookrightarrow L((i-1)\vec{x}_3)$, we get

$$(4.6) \qquad\qquad\qquad \mathrm{Ext}^1(L((i-1)\vec{x}_3), X) \neq 0.$$

Next, we claim that $\mathrm{Hom}(E_i, X) = 0$. Otherwise, we have $X \in \mathcal{S}(E_i \to)$. Notice that $L((i-1)\vec{x}_3) \in \mathcal{S}(E_i \to)$, hence X and $L((i-1)\vec{x}_3)$ belong to the same slice. It follows that $X \oplus L((i-1)\vec{x}_3)$ is extension-free, contradicting to (4.6), as claimed. So by definition, we get $\tau^{-1} X \in \mathrm{Dom}^+(E_i) \subseteq \mathrm{Dom}^+(E_j)$. Then by Serre duality and Lemma 3.13,

$$\mathrm{Hom}(E_j^d, X) = D\,\mathrm{Ext}^1(\tau^{-1} X, E_j^d) = 0.$$

Moreover, for any $i \leq k \leq j$, $\tau^{-1} X \in \mathrm{Dom}^+(E_i) \subseteq \mathrm{Dom}^+(E_k)$ implies that $\mathrm{Ext}^1(\tau^{-1} X, E_k) = 0$. It follows that $\mathrm{Hom}(E_k, X) = 0$. Hence $\mathrm{Hom}(T, X) \neq 0$ implies $\mathrm{Hom}(E_i^u, X) \neq 0$. Then there exists some $0 \leq m \leq i-1$, such that

$$(4.7) \qquad\qquad\qquad \mathrm{Hom}(L(m\vec{x}_3), X) \neq 0.$$

By applying $\mathrm{Hom}(-, X)$ to the exact sequence

$$0 \to L \to L(m\vec{x}_3) \to S \to 0,$$

where S is a coherent sheaf of finite length, we get an exact sequence

$$0 \to \mathrm{Hom}(S, X) \to \mathrm{Hom}(L(m\vec{x}_3), X) \to \mathrm{Hom}(L, X).$$

Since $\mathrm{Hom}(S, X) = 0$, (4.7) induces that

$$\mathrm{Hom}(L, X) \neq 0.$$

This finishes the proof. $\qquad\square$

Lemma 4.6. *The category $\mathscr{C}_1$ doesn't contain any line bundle.*

Proof. For contradiction we assume $L(\vec{x}) \in \mathscr{C}_1$ for some $\vec{x} \in \mathbb{L}$. Then by (4.5),

$$\operatorname{Hom}(L, L(\vec{x})) \neq 0 \quad \text{and} \quad \operatorname{Ext}^1(L((i-1)\vec{x}_3), L(\vec{x})) \neq 0.$$

It follows from (2.3) and (3.2) that

$$\vec{x} \geq 0 \quad \text{and} \quad \vec{x} \leq \vec{\omega} + (i-1)\vec{x}_3 \leq \vec{\omega} + \vec{c},$$

which is a contradiction to (2.2), this finishes the proof. $\qquad\square$

Denote by $\mathscr{L}$ the set of all line bundles in $\operatorname{coh}\mathbb{X}$ and $\underline{\operatorname{vect}}\mathbb{X} = \operatorname{vect}\mathbb{X}/[\mathscr{L}]$ the stable category of vector bundles obtained by factoring out all line bundles. Then Lemma 4.6 shows that the "missing part" $\mathscr{C}_1$ is a subcategory of $\underline{\operatorname{vect}}\mathbb{X}$. For simplification, we denote $\operatorname{Hom}_{\underline{\operatorname{vect}}\mathbb{X}}(-,-)$ by $\underline{\operatorname{Hom}}(-,-)$.

Lemma 4.7. *Let X, Y, Z be indecomposable vector bundles satisfying*

$$\underline{\operatorname{Hom}}(X, Z) \neq 0 \quad \text{and} \quad \underline{\operatorname{Hom}}(Z, Y) \neq 0.$$

Then $X, Y \in \mathscr{C}_1$ imply $Z \in \mathscr{C}_1$.

Proof. Since $\underline{\operatorname{Hom}}(X, Z) \neq 0$, there exists a non-zero morphism $\phi : X \to Z$ in $\operatorname{coh}\mathbb{X}$, which can not factor through line bundles. It follows that $\operatorname{Im}\phi = X$, that is, ϕ is injective. Since $X \in \mathscr{C}_1$, we have $\operatorname{Hom}(L, X) \neq 0$. Applying $\operatorname{Hom}(L, -)$ to ϕ, it follows that

$$\operatorname{Hom}(L, Z) \neq 0.$$

Dually, $\underline{\operatorname{Hom}}(Z, Y) \neq 0$ and $Y \in \mathscr{C}_1$ imply that

$$\operatorname{Ext}^1(L((i-1)\vec{x}_3), Z) \neq 0.$$

Thus by (4.5), $Z \in \mathscr{C}_1$. $\qquad\square$

For any two vector bundles X, Y, denote by $[\mathscr{L}](X, Y)$ the ideal of all morphisms from X to Y in $\operatorname{coh}\mathbb{X}$ which factor through a finite direct sum of line bundles. The following proposition indicates that $\mathscr{C}_1$ is actually a full subcategory of $\underline{\operatorname{vect}}\mathbb{X}$.

Proposition 4.8. *For any two vector bundles $X, Y \in \mathscr{C}_1$, we have*

$$\mathscr{C}_1(X, Y) = \operatorname{Hom}(X, Y)/[\mathscr{L}](X, Y).$$

Proof. Obviously, we have a surjection $\pi : \operatorname{Hom}(X, Y) \twoheadrightarrow \mathscr{C}_1(X, Y)$. By Lemma 4.6, we have $[\mathscr{L}](X, Y) \subseteq \operatorname{Ker}\pi$. On the other hand, for any $f \in \operatorname{Hom}(X, Y)$ with $\pi(f) = 0$, we know that f factors through some object $Z \in \mathcal{X}_0 \cup \mathcal{X}_1$. We claim that f can factor through a direct sum of line bundles. Otherwise, there exists an indecomposable summand Z_k of Z with $\operatorname{rk} Z_k = 2$ satisfying

$$\underline{\operatorname{Hom}}(X, Z_k) \neq 0 \quad \text{and} \quad \underline{\operatorname{Hom}}(Z_k, Y) \neq 0.$$

Then by Lemma 4.7, $Z_k \in \mathscr{C}_1$, a contradiction. It follows that $f \in [\mathscr{L}](X, Y)$. Hence $\operatorname{Ker}\pi = [\mathscr{L}](X, Y)$, this finishes the proof. $\qquad\square$

Lemma 4.9. *The category $\mathscr{C}_1$ is an abelian category.*

Proof. For any indecomposable object $X \in \mathscr{C}_1$, applying $\operatorname{Hom}(-, X)$ to the injective map $L((i-1)\vec{x}_3) \hookrightarrow L(\vec{c})$, we get $\operatorname{Ext}^1(L(\vec{c}), X) \neq 0$. Hence $X \in \mathscr{C}_L$. That is, $\mathscr{C}_1$ is a subcategory of $\mathscr{C}_L$. Moreover, for any vector bundles $X, Y \in \mathscr{C}_1$, by the same proof of Proposition 4.8, we know that

$$\mathscr{C}_L(X, Y) = \operatorname{Hom}(X, Y)/[\mathscr{L}](X, Y) = \mathscr{C}_1(X, Y).$$

Hence $\mathscr{C}_1$ is a full subcategory of $\mathscr{C}_L$.

We are going to show that $\mathscr{C}_1$ is closed under extension, kernel of surjective and cokernel of injective in $\mathscr{C}_L$. Then $\mathscr{C}_1$ inherits the abelianness of $\mathscr{C}_L$.

Firstly, we show that $\mathscr{C}_1$ is closed under extension in $\mathscr{C}_L$. Let

$$0 \to X \to \oplus_k Z_k \to Y \to 0$$

be a non-split exact sequence in $\mathscr{C}_L$ with X, Y from $\mathscr{C}_1$. Without loss generality, we assume X, Y are indecomposable. We need to show each $Z_k \in \mathscr{C}_1$. It suffices to show that

(4.8) $$\mathrm{Ext}^1(L((i-1)\vec{x}_3), Z_k) \neq 0.$$

There are two cases to consider:

Case 1: $\mathscr{C}_L(Z_k, Y) \neq 0$, then $\underline{\mathrm{Hom}}(Z_k, Y) \neq 0$ follows. By the proof of Lemma 4.7, each non-zero morphism $\phi : Z_k \to Y$ is injective in $\mathrm{coh}\,\mathbb{X}$. Assume ϕ fits into the following exact sequence in $\mathrm{coh}\,\mathbb{X}$:

(4.9) $$0 \to Z_k \xrightarrow{\phi} Y \to S \to 0,$$

where S is a coherent sheaf of finite length since both Z_k and Y are of rank two. Notice that $Y \in \mathscr{C}_1$, it follows that

(4.10) $$\mathrm{Ext}^1(L((i-1)\vec{x}_3), Y) \neq 0.$$

Now by applying $\mathrm{Hom}(L((i-1)\vec{x}_3), -)$ to (4.9), we get an exact sequence $\mathrm{Ext}^1(L((i-1)\vec{x}_3), Z_k) \to \mathrm{Ext}^1(L((i-1)\vec{x}_3), Y) \to \mathrm{Ext}^1(L((i-1)\vec{x}_3), S)$. Note that $\mathrm{Ext}^1(L((i-1)\vec{x}_3), S) = 0$. Then (4.8) follows from (4.10).

Case 2: $\mathscr{C}_L(Z_k, Y) = 0$, then $X \to Z_k$ is surjective in $\mathscr{C}_L$. Notice from [2] that $\mathscr{C}_L$ is equivalent to the module category of $k\overrightarrow{A_{n-1}}$. Hence from the structure of $\mathscr{C}_L$, we know that in $\Gamma(\mathrm{vect}\,\mathbb{X})$,

(4.11) $$Z_k \in \mathcal{S}(X \to) \quad \text{and} \quad l(Z_k) \leq l(X).$$

Since $X \in \mathscr{C}_1$, it follows that $\mathrm{Ext}^1(L((i-1)\vec{x}_3), X) \neq 0$. Then by Serre duality,

(4.12) $$\mathrm{Hom}(X, \tau L((i-1)\vec{x}_3)) \neq 0.$$

Hence there exists some $m \geq 1$, such that $\tau^m L((i-1)\vec{x}_3) \in \mathcal{S}(X \to)$. Combining with (4.11), and noticing that $\tau^m L((i-1)\vec{x}_3)$ is of length one, we get $\tau^m L((i-1)\vec{x}_3) \in \mathcal{S}(Z_k \to)$. Hence $\mathrm{Hom}(Z_k, \tau^m L((i-1)\vec{x}_3)) \neq 0$. According to Corollary 3.4,

$$\mathrm{Hom}(Z_k, \tau L((i-1)\vec{x}_3)) \neq 0.$$

By Serre duality, (4.8) holds, as claimed.

Secondly, assume $\phi : X \to Y$ is surjective in $\mathscr{C}_L$ with $X, Y \in \mathscr{C}_1$. If ϕ is isomorphism, then $\mathrm{Ker}\,\phi = 0 \in \mathscr{C}_1$. Otherwise, for any indecomposable direct summand Z_k of $\mathrm{Ker}\,\phi$, there exists an non-zero morphism from Z_k to X in $\mathscr{C}_L$. By the same proof of *Case* 1 above, $X \in \mathscr{C}_1$ implies that $Z_k \in \mathscr{C}_1$.

Thirdly, assume $\phi : X \to Y$ is injective in $\mathscr{C}_L$ with $X, Y \in \mathscr{C}_1$. If ϕ is isomorphism, then $\mathrm{Coker}\,\phi = 0 \in \mathscr{C}_1$. Otherwise, for any indecomposable direct summand Z_k of $\mathrm{Coker}\,\phi$, $Y \to Z_k$ is surjective. By similar arguments of *Case* 2 above, $Y \in \mathscr{C}_1$ implies that $Z_k \in \mathscr{C}_1$. This finishes the proof. $\square$

Similarly, we have

Lemma 4.10. *Assume* $\mathcal{S}(E_j \to) \cap \mathcal{L}_{n+1} = L'$, *then*

$$\mathscr{C}_2 = \mathrm{add}\{X \in \mathrm{ind}(\mathrm{coh}\,\mathbb{X}) | \mathrm{Hom}(L'(-(n-j+1)\vec{x}_3), X) \neq 0 \neq \mathrm{Ext}^1(L', X)\}.$$

Moreover, $\mathscr{C}_2$ *is an abelian category.*

Now we give our main result in this section.

Theorem 4.11. *Let T be a tilting bundle of the form (4.3) and $\mathscr{C}$ be the corresponding "missing part". Assume $\mathcal{S}(\to E_i) \cap \mathcal{L}_1 = L$ and $\mathcal{S}(E_j \to) \cap \mathcal{L}_{n+1} = L'$. Then $\mathscr{C}$ can be decomposed to a product of two abelian categories*

$$\mathscr{C} = \mathscr{C}_1 \coprod \mathscr{C}_2,$$

where

$$\mathscr{C}_1 = \mathrm{add}\{X \in \mathrm{ind}(\mathrm{coh}\,\mathbb{X}) | \mathrm{Hom}(L, X) \neq 0 \neq \mathrm{Ext}^1(L((i-1)\vec{x}_3), X)\},$$

and

$$\mathscr{C}_2 = \mathrm{add}\{X \in \mathrm{ind}(\mathrm{coh}\,\mathbb{X}) | \mathrm{Hom}(L'(-(n-j+1)\vec{x}_3), X) \neq 0 \neq \mathrm{Ext}^1(L', X)\}.$$

Proof. Combine with Lemma 4.4, Lemma 4.5, Lemma 4.9 and Lemma 4.10 to finish the proof. $\square$

For other choices of $T^+(E_i) \in \mathrm{Br}^+(E_i)$ and $T^-(E_j) \in \mathrm{Br}^-(E_j)$, by similar arguments one can obtain the similar decomposition of $\mathscr{C}$ with some modifications of L and L'. Summarize up, we have

Theorem 4.12. *Let T be a tilting bundle of the form (3.18) and $\mathscr{C}$ be the corresponding "missing part". Then $\mathscr{C}$ can be decomposed to a product of two abelian categories.*

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